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Integrability vs. Information Loss: A Simple Example

Vijay Balasubramanian*, Bartłomiej Czech†, Klaus Larjo‡ and Joan Simón§,

*David Rittenhouse Laboratories, University of Pennsylvania,
Philadelphia, PA 19104, U.S.A.*

Abstract

The half-BPS sector of Yang-Mills theory with 16 supercharges is integrable: there is a set of commuting conserved charges, whose eigenvalues can completely identify a state. We show that these charges can be measured in the dual gravitational description from asymptotic multipole moments of the space-time. However, Planck scale measurements are required to separate the charges of different microstates. Thus, semiclassical observers making coarse-grained measurements necessarily lose information about the underlying quantum state.

*vijay@physics.upenn.edu

†czech@sas.upenn.edu

‡klarjo@physics.upenn.edu

§jsimon@bokchoy.hep.upenn.edu

1 Introduction

The half-BPS sector of $SU(N)$ Yang-Mills theory with 16 supercharges contains a complete commuting set of N conserved charges M_k , and an eigenstate of these M_k can be identified from the eigenvalues. The gravitational duals of these states in AdS_5 have also been identified [1], and it has been argued that typical half-BPS states behave universally like an extremal black hole in response to almost all probes [2]. This implies that information about the underlying microstate is lost. Here we show how this tension between integrability and information loss is resolved. We demonstrate that angular moments of the gravitational solution that can be read off from the asymptotic metric directly measure the higher conserved charges of the underlying quantum microstate. The low moments have magnitudes large enough for semiclassical observation, but measuring the differences in these moments between typical states requires Planck scale precision. The high moments vary strongly between typical states, but their magnitudes are so small that Planck scale precision is again required to measure them. Thus, the coarse-grained semiclassical gravitational description of the underlying exact quantum microstate necessarily loses information.

2 Half-BPS states

2.1 Description in gauge theory

$SU(N)$ Yang-Mills theory on $S^3 \times R$ with 16 supercharges has a spectrum of half-BPS states. The highest weight representatives in each BPS multiplet are created by operators that are gauge invariant polynomials in the zero-modes of a single adjoint scalar field. The Hamiltonian of the gauge theory restricted to these states reduces to that of a Hermitian matrix harmonic oscillator [3, 4]. This is solved by going to the eigenvalue basis, and redefining fields to arrive at a theory of N free fermions in a harmonic potential. A basis of highest weight half-BPS states is completely specified by a set of increasing integers $\mathcal{F} = \{f_1, f_2, \dots, f_N\}$ that provide the excitation numbers of each individual fermion $E_i = \hbar(f_i + \frac{1}{2})$, $i = 1, \dots, N$. Equivalently, we can reproduce the state by giving a set of non-decreasing integers $\{r_i\}$, which measure the excitation of any given fermion above the vacuum

$$r_i = \frac{E_i}{\hbar} - i + \frac{1}{2}. \quad (1)$$

This data can be encoded as a Young diagram in which the i -th row has length r_i .

The energies $\{f_1, \dots, f_N\}$ completely specify these basis states. The states can also be specified in terms of the moments

$$M_k = \sum_{i=1}^N f_i^k = \text{Tr}(H_N^k / \hbar^k) \quad ; \quad k = 0, \dots, N, \quad (2)$$

where H_N is the Hamiltonian acting on the N fermion Hilbert space with the zero point energy removed. Manifestly, the M_k are conserved charges of the system of

fermions in a harmonic potential [5]. The basis of states with fixed fermion excitation energies that was described above consists of eigenstates of the moment operators. The individual excitation energies \mathcal{F} in these eigenstates can always be reconstructed from the set of moments $\mathcal{M} = \{M_0, M_1, \dots, M_N\}$. To see this, construct the characteristic polynomial

$$P = \det(x \mathbf{I}_N - H_N/\hbar) = \prod_{i=1}^N (x - f_i). \quad (3)$$

This product can be expanded in symmetric products of the f_i as

$$P = \sum_{p=0}^N (-1)^p \pi_p x^{N-p} \quad ; \quad \pi_k = \sum_{i_1 < i_2 < \dots < i_k} f_{i_1} f_{i_2} \cdots f_{i_k}. \quad (4)$$

The π_i are given recursively in terms of the moments M_k by the Newton–Girard formula

$$m\pi_m + \sum_{k=1}^m (-1)^k M_k \pi_{m-k} = 0. \quad (5)$$

Thus, given a measurement of the moments M_k for $k = 1, \dots, N$ one can compute the symmetric products π_i and from these determine the fermion excitations \mathcal{F} that completely determine the basis BPS states as the roots of the characteristic polynomial (3).

Very heavy half-BPS states ($M_1 = \text{Tr}(H_N/\hbar) \sim N^2$) have a dual description in terms of large-scale classical solutions of the maximally supersymmetric gravity in AdS_5 . In [2] it was shown that almost all such eigenstates of the moment operators $\{M_k\}$ lie very close to a certain “typical state” with a characteristic distribution of fermion energies. In an ensemble of states in which the maximum excitation energy of any given fermion is bounded (so that $r_i \leq N_c \forall i$) the expected excitation energies of the fermions in the typical state are given by:

$$\langle r_i \rangle = \sum_{j=0}^{i-1} \frac{e^{-\beta(N-j)-\lambda}}{1 - e^{-\beta(N-j)-\lambda}} \quad (6)$$

where β is an “inverse temperature” used to fix the total energy and λ is a “chemical potential” fixing the maximum excitation of individual fermions. The semiclassical limit for this system is obtained by sending $\hbar \rightarrow 0$ with $N\hbar$ kept constant to fix the Fermi level. In this limit, almost all half BPS states have fermion energy distributions that are small fluctuations around (6). In this limit we can treat the fermion number i and the excitation energies r_i as continuous variables and relabel

$$i \rightarrow x \text{ and } r_i \rightarrow y(x). \quad (7)$$

Any eigenstate of the moment operators that has a semiclassical limit can then be described in terms of such a function $y(x)$. In particular, the typical state (6) is summarized as a limit curve

$$C(N, N_c) e^{-\beta(N-x)} + D(N, N_c) e^{-\beta(y-N_c)} = 1, \quad (8)$$

where C and D are determined by fixing the total energy and maximum excitation. The maximum entropy results when $\beta \rightarrow 0$ in which case this limit curve simply reduces to

$$y = \frac{N_c}{N} x \quad (9)$$

Thus, in the large N, N_c limit, almost all half-BPS states with bounded fermion excitations lie very close to the line (9). Said otherwise, the *uniform* distribution over all sets of integers $r_1 \leq r_2 \leq \dots \leq r_N \leq N_c$ is strongly localized on partitions that lie close to the curve (9). We will have use of this fact later.

These half-BPS states can also be described in terms of a distribution of fermions on the single particle harmonic oscillator phase space (p, q) [2]. Wigner [6] described a distribution function that encodes the expectation value of Weyl (symmetric) ordered quantum observables,

$$\int dp dq W(p, q) f(p, q) = \langle \mathcal{O}_W(f) \rangle, \quad (10)$$

where \mathcal{O}_W is the Weyl-ordered operator corresponding to the classical function f . The Wigner distribution corresponding to a state \mathcal{F} was computed in [2] to be

$$W(r) = \frac{1}{\pi \hbar} e^{-\frac{r^2}{\hbar}} \sum_{f \in \mathcal{F}} (-1)^f L_f \left(\frac{2r^2}{\hbar} \right), \quad (11)$$

where $L_f(x)$ is a Laguerre polynomial and $r^2 = q^2 + p^2$. Since we are considering an eigenstate of the Hamiltonian, the corresponding phase space distribution is rotationally invariant. If a state is an eigenstate of the M_k , all of these moments can be extracted from the Wigner distribution. Two other interesting distributions on phase space were identified in [2]. The Husimi distribution

$$\text{Hu}(r) = \frac{1}{2\pi \hbar} e^{-\frac{r^2}{\hbar}} \sum_{f \in \mathcal{F}} \frac{1}{f!} \left(\frac{r^2}{\hbar} \right)^f \quad (12)$$

arises by smoothing the Wigner distribution with a Gaussian kernel at the \hbar scale, and computes the expectation values of reverse normal ordered operators [7]. Finally, consider a state that is described in the semiclassical $\hbar \rightarrow 0$ limit by a limit curve $y(x)$ as in (7, 8). It was shown in [2] that a semiclassical observer of such a state, having access only to areas larger than \hbar in phase space, interacts with the effective “grayscale distribution”

$$g(r) = \frac{1}{1 + dy/dx}. \quad (13)$$

2.2 Description in gravity

By analyzing symmetries, the half-BPS states described above should be dual in the semiclassical limit to solutions of IIB supergravity with $SO(4) \times SO(4) \times U(1)$

symmetry with 5-form flux and constant dilaton. These solutions have been found in [1]:

$$ds^2 = -h^{-2} (dt + V_i dx^i)^2 + h^2 (dy^2 + dx^i dx^i) + R^2 d\Omega_3^2 + \tilde{R}^2 d\tilde{\Omega}_3^2, \quad (14)$$

where the coefficients are given in terms of a function $u(x^1, x^2, y)$ as

$$R^2 = y \sqrt{\frac{1-u}{u}}, \quad \tilde{R}^2 = y \sqrt{\frac{u}{1-u}}, \quad h^{-2} = \frac{y}{\sqrt{u(1-u)}}. \quad (15)$$

The one form V is

$$V_i(x_1, x_2, y) = -\frac{\epsilon_{ij}}{\pi} \int_{\mathbb{R}^2} \frac{u(x'_1, x'_2, 0) (x_j - x'_j) dx'_1 dx'_2}{[(\vec{x} - \vec{x}')^2 + y^2]^2}. \quad (16)$$

Thus, the function u completely specifies the solution. This function in turn satisfies a harmonic equation in y and as such is fully determined by its boundary condition in the $y = 0$ plane

$$u(r, \varphi, y) = \frac{y^2}{\pi} \int_{\mathbb{R}} \frac{u(r', \varphi', 0) d^2 \vec{r}'}{[(\vec{r} - \vec{r}')^2 + y^2]^2}, \quad (17)$$

where we have used polar coordinates for the (x^1, x^2) plane. The full solution also contains a 5-form field strength that we are omitting here.

A dictionary between these solutions and the half-BPS field theory states has been established in [1, 2]. The (x_1, x_2) plane at $y = 0$ is identified with the single particle oscillator phase space in the dual gauge theory:

$$(x_1, x_2) \leftrightarrow (p, q). \quad (18)$$

The Planck scale l_p is related to \hbar in the field theory:

$$l_p^4 \leftrightarrow \hbar. \quad (19)$$

Finally, the boundary condition function $u(x^1, x^2, 0)$ is identified with the single particle phase space distribution in the fermionic description of half-BPS states in the field theory. The eigenstates of M_k that we are interested in are all rotationally invariant in phase space, so, taking

$$r^2 = (x^1)^2 + (x^2)^2 \leftrightarrow p^2 + q^2 \quad (20)$$

and writing $u(x^1, x^2, 0)$ as $u(r)$ we could identify

$$u(r) \leftrightarrow 2\pi\hbar W(r), \quad 2\pi\hbar Hu(r), \quad 2\pi g(r) \quad (21)$$

We will see in Sec. 3.2 that a classical observer will not be able to detect the differences between these choices.

3 Integrability and the gravitational solution

Above we saw that a half BPS state in field theory with a fixed set of fermion excitation energies \mathcal{F} can be completely identified by a measurement of a set of gauge-invariant observables, the moments \mathcal{M} . However, the gravitational description of half BPS states involves the effective single particle phase space distribution (21) and as such appears to lose information about the N fermion excitation energies that are necessary to characterize the state completely. Here we show that for eigenstates of the moment operators M_k , no information is lost. All the moments of the fermion energies are stored in angular moments of the gravitational solution that can be measured from infinity. In this way the gravitational solution preserves the integrable character of half BPS states.

3.1 Multipole expansion and integrable charges

At infinity, a natural tool for studying the state of a spacetime is the analysis of the multipole moments of all fields. Such an expansion is usually done at the level of observable quantities, but since all half BPS geometries are characterized by the scalar function $u(x^1, x^2, y)$, it suffices to study the asymptotic multipoles of u . We choose the boundary condition for u on the $y = 0$ plane as corresponding to the Wigner distribution on the fermion phase space of the dual field theory:

$$u(r) = 2\pi\hbar W(r) \quad (22)$$

following (21). Then, recalling the exact expression (11) for the Wigner distribution associated to a half-BPS eigenstate of the M_k , we expand the denominator of (17) as a power series in $\frac{r'^2 - 2\vec{r} \cdot \vec{r}'}{r^2 + y^2}$. The integrals can be done explicitly using

$$\begin{aligned} \int_0^\infty \int_0^{2\pi} dr d\varphi e^{-r^2} L_f(2r^2) & r^{l+1} (n_1 \cos \varphi + n_2 \sin \varphi)^p \\ &= \frac{\pi f!}{2^p} \binom{p}{\frac{p}{2}} \sum_{k=0}^f \frac{(-1)^k 2^k (k + \frac{l}{2})!}{(f-k)!(k!)^2} \delta_{p,\text{even}}, \end{aligned} \quad (23)$$

where $|\vec{n}|^2 = 1$. Spacelike infinity is reached in the solutions (14) by either going to large radial distances in the (x^1, x^2) plane at $y = 0$ or by going to large y . Thus it is natural, at infinity, to introduce the new radial coordinate ρ :

$$r = \rho \sin \theta, \quad y = \rho \cos \theta, \quad \text{with } \theta \in [0, \frac{\pi}{2}]. \quad (24)$$

In this coordinate system the boundary of the spacetime lies at $\rho \rightarrow \infty$. The new angular variable θ , which measures the angle between the two radial variables $\{r, y\}$ becomes the azimuthal angle in the 5-sphere of the asymptotic $AdS_5 \times S^5$ geometry.

After some manipulation we find that in this coordinate system u has the asymptotic multipole expansion

$$u(\rho, \theta) = 2 \cos^2 \theta \sum_{l=0}^{\infty} \frac{\hbar^{l+1} \sum_{f \in \mathcal{F}} A^l(f)}{\rho^{2l+2}} (-1)^l (l+1) {}_2F_1(-l, l+2, 1; \sin^2 \theta), \quad (25)$$

where ${}_2F_1$ is the hypergeometric function and $A^l(f)$ is a polynomial of order l in f :

$$A^n(f) \equiv \sum_{s=0}^f \frac{(-1)^{f-s} 2^s f!}{(f-s)! s!} (s+1)_n. \quad (26)$$

The Pochhammer symbol is defined by $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{(\alpha+n-1)!}{(\alpha-1)!}$. Here $\hbar \leftrightarrow l_p^4$ as in (19). All of these sums can be computed by introducing the generating function

$$B(n, f, a) = \sum_{s=0}^f \binom{f}{s} (-1)^{f-s} a^{s+n} = a^n (a-1)^f, \quad (27)$$

from which we can derive

$$A^n(f) = \left(\frac{d}{da} \right)^n B(n, f, a) \Big|_{a \rightarrow 2}. \quad (28)$$

The first few sums are

$$\begin{aligned} A^0(f) &= 1, \\ A^1(f) &= 2f + 1, \\ A^2(f) &= (2f+1)^2 + 1, \\ A^3(f) &= (2f+1)^3 + 5(2f+1), \\ A^4(f) &= (2f+1)^4 + 14(2f+1)^2 + 9, \\ A^5(f) &= (2f+1)^5 + 30(2f+1)^3 + 89(2f+1), \end{aligned}$$

These are even and odd in $(2f+1)$ as can be seen from the recurrence relation

$$A^{n+1}(f) = (2f+1)A^n(f) + n^2 A^{n-1}(f), \quad (29)$$

that allows us to compute any such sum. Another convenient form is

$$A^n(f) = \left(\frac{d}{da} \right)^n B(n, f, a) \Big|_{a \rightarrow 2} = n! \sum_{k=0}^n \binom{n}{k} \binom{f}{k} 2^k. \quad (30)$$

Since the whole ten dimensional geometry is determined by the function u , (25) permits computation of all the angular moments of the metric. This is given in Appendix A. For example, the asymptotic multipole expansion of the V_1 component of the V_i shift vector in (14) is

$$\begin{aligned} V_1(r, \varphi, \theta) &= -2 \sin \varphi \sin \theta \left(\frac{\hbar N}{\rho^3} + \sum_{l=1}^{\infty} \frac{\hbar^{l+1} \sum_{f \in \mathcal{F}} A^l(f)}{\rho^{2l+3}} (-1)^l (l+1) \cdot \right. \\ &\quad \left. \cdot \left[{}_2F_1(-l, l+2, 1; \sin^2 \theta) + l {}_2F_1(1-l, l+2, 2; \sin^2 \theta) \right] \right). \end{aligned} \quad (31)$$

The angular moments of curvature invariants can be computed from this data.

Measuring the charges: In this multipole expansion, the data about the underlying state $\{\mathcal{F}\}$ enters the l^{th} moment in sums of the form

$$\sum_{f \in \mathcal{F}} A^l(f) = \sum_{k=0}^l c_k M_k \quad (32)$$

where the M_k are the moments defined in (2) and c_k is the coefficient of f^k in the polynomial expansion of $A^l(f)$. Thus a measurement of the first N multipole moments of the metric functions can be inverted to give the set of charges \mathcal{M} of the underlying state, from which the complete wavefunction can be reconstructed.

3.2 The semiclassical limit

The semiclassical limit for half BPS states is

$$\hbar \rightarrow 0 \quad ; \quad \hbar N = \alpha = \text{fixed} . \quad (33)$$

Since the moments M_l scale with N as

$$M_l = m_l N^{l+1} \quad (34)$$

we see that as $\hbar \rightarrow 0$

$$\hbar^{l+1} M_l \rightarrow m_l \alpha^{l+1} . \quad (35)$$

Thus, in the semiclassical limit the multipole expansion reduces to

$$u(\rho, \theta) = 2 \cos^2 \theta \sum_{l=0}^{\infty} \frac{2^l \alpha^{l+1} m_l}{\rho^{2l+2}} (-1)^l (l+1) {}_2F_1(-l, l+2, 1; \sin^2 \theta) . \quad (36)$$

The l^{th} multipole is dominated by M_l with subleading corrections from the lower moments. Thus, in the semiclassical limit, the amplitude of each multipole moment is directly related to a higher order conserved charge of the underlying integrable system.

The computations above were carried out by identifying the boundary condition for the u function in the $y = 0$ plane with the Wigner distribution in the harmonic oscillator phase space (22). However, there are other distributions on phase space which differ in the ordering prescription that they assume for quantum mechanical operators. Any of these distributions is a candidate for identification with the u function and it was proposed in [2] and each choice should be appropriate for describing the effective metric sensed by different quantum gravity observables. However, since ordering prescriptions only lead to differences in observables at higher orders in \hbar , the leading semiclassical result (36) should be universal. We demonstrate this below for the Husimi distribution (12) and for the grayscale distribution (13).

Husimi distribution: We can carry out the multipole expansion of Sec. 3.1 after identifying the boundary condition for the u function on the $y = 0$ plane as

$$u(r) = 2\pi\hbar \text{Hu}(r) \quad (37)$$

where the Husimi distribution $\text{Hu}(r)$ for a given set of fermion excitation energies \mathcal{F} is given in (12). We obtain

$$u_H(\rho, \theta) = 2\cos^2\theta \sum_{l=0}^{\infty} \frac{2^l \hbar^{l+1} \sum_{f \in \mathcal{F}} \frac{(f+l)!}{f!}}{\rho^{2l+2}} (-1)^l (l+1) {}_2F_1(-l, l+2, 1; \sin^2\theta). \quad (38)$$

Since $\frac{(f+l)!}{f!} = f^l + (\text{lower powers of } f)$, the semiclassical limit gives

$$u_H^{s.c.} \rightarrow 2\cos^2\theta \sum_{l=0}^{\infty} \frac{2^l \alpha^{l+1} m_l}{\rho^{2l+2}} (-1)^l (l+1) {}_2F_1(-l, l+2, 1; \sin^2\theta). \quad (39)$$

This agrees exactly with the semiclassical limit in (36). Thus the semiclassical asymptotic observer will always measure the same multipole moments. However, the sub-leading terms that are suppressed in powers of \hbar (or, equivalently, l_p , see (19)) differ between the geometries based on the Wigner and Husimi distributions. Presumably, this implies that while the classical spacetime does not depend on the operator ordering prescription, the underlying ‘‘quantum foam’’ [2] looks different to different quantum mechanical observables.

Grayscale distribution: Finally we can identify the boundary condition for u with the grayscale distribution (13) as

$$u(r) = 2\pi g(r). \quad (40)$$

In this case it is difficult to analyze the multipole expansion in complete generality. Thus we present the computation for a state whose semiclassical limit curve (see (7)) is

$$y(x) = (\delta - 1)x \quad ; \quad x \in (0, N-1) \quad (41)$$

This limit curve corresponds to the extremal superstar spacetime (see [8, 2] for details). The corresponding grayscale phase space distribution is (13)

$$u = \frac{1}{1+y'} = \frac{1}{\delta}, \quad 0 \leq r \leq r_0, \quad (42)$$

where $r_0 = \sqrt{2\hbar\delta N}$ to leading order in N . Because the curve (41) is entirely determined by δ , the moments M_k are all expressible in terms of this parameter:

$$M_k = \sum_{i=1}^N f_i^k = \sum_{i=0}^{N-1} (\delta i)^k = \delta^k \sum_{i=0}^{N-1} i^k \approx \delta^k \frac{N^{k+1}}{k+1}. \quad (43)$$

Then we can once again compute the asymptotic expansion and find that in the semiclassical limit

$$u_G = 2 \cos^2 \theta \sum_{l=0}^{\infty} \frac{2^l \alpha^{l+1} m_l}{\rho^{2l+2}} (-1)^l (l+1) {}_2F_1(-l, l+2, 1; \sin^2 \theta), \quad (44)$$

where $m_l = \frac{\delta^l}{l+1}$. This matches with the Wigner (36) and Husimi (39) geometries as expected.

4 Semiclassical observers and information loss

Above we showed that the entire tower of conserved charges of half BPS states is stored, in the gravitational description, in multipole moments of the spacetime. We will now show that a semiclassical observer will not be able to access this information. First, we will argue that such an observer, with access to length scales longer than the Planck length, will only be able to measure the very low multipoles. Second, we will show that these low multipoles are essentially universal for almost all states. Thus, semiclassical observers will necessarily lose information even though it is present in the full theory.

4.1 High multipoles cannot be measured

Recall that the semiclassical limit for half BPS states is $\hbar \rightarrow 0$ with $\hbar N = \alpha$ fixed (33). Translating this into gravity using $l_p^4 \leftrightarrow \hbar$ this amounts to

$$\hbar N \leftrightarrow l_p^4 N \sim g_s l_s^4 N \sim L^4 = \alpha = \text{fixed} \quad ; \quad L \sim l_s (g_s N)^{1/4} \quad (45)$$

where l_s is the string length, g_s is the string coupling, and L is the length scale associated to the asymptotic $\text{AdS}_5 \times S^5$ spacetime using the standard AdS/CFT dictionary. Thus, the semiclassical limit (33) that we have been using is the same as the standard limit in the AdS/CFT correspondence, namely $g_s \rightarrow 0, N \rightarrow \infty$ with L fixed.

One way of measuring the l^{th} multipole in (25,36) is to compute the $(2l)^{\text{th}}$ derivative of the metric functions or any suitable invariant constructed from them. Consider an apparatus of finite size λ that makes such a measurement. In order to compute the k^{th} derivative of a quantity within a region of size λ , the apparatus will have to make measurements at a scale λ/k . However, a semiclassical apparatus can only measure quantities over distances larger than the Planck length. Thus, the k^{th} derivative can only be measured if

$$\frac{\lambda}{k} > l_p = g_s^{1/4} l_s \quad (46)$$

Setting the size of the apparatus to be a fixed multiple of the AdS scale

$$\lambda = \gamma L, \quad (47)$$

this says that

$$k < \gamma N^{1/4} \quad (48)$$

for a derivative to be semiclassically measurable. In order to identify the underlying quantum state we have shown that $O(N)$ multipoles must be measured. Since $N^{1/4}/N \rightarrow 0$ as $N \rightarrow \infty$ we see that the semiclassical observer has access to a negligible fraction of the information needed to identify the quantum state.

In order to measure the first N multipoles *without* making Planckian measurements, an observer would require an apparatus of size $\lambda \sim N^{3/4}L$. But in the semiclassical limit, this size diverges. This is an interesting appearance of the connection in gravity between the extreme UV and the extreme IR.

We might attempt to avoid the difficulty of probing high order derivatives in spacetime by directly measuring the angular moments of (25,36) along the θ direction. Consider a location where this circle has a circumference λ . Then to measure the k^{th} multipole, an apparatus will have to measure the amplitude of a fluctuation along this circle which has $O(k)$ nodes. For a semiclassical apparatus, the spacing between nodes must be bigger than Planck length in order to be measurable. This requires $\lambda/k > l_p$ just as in (46). Then by reasoning identical to the above, only $O(N^{1/4})$ multipoles will be measurable. One might attempt to increase the number of measurable multipoles by moving out to locations at which the θ circle has a large circumference. However, as described below (24), θ is the azimuthal coordinate in the S^5 part of the asymptotic $\text{AdS}_5 \times S^5$ in the geometry. As such its size remains of $O(L)$ even at infinity, and direct measurement of higher multipoles remains impossible for the semiclassical observer.

4.2 Low multipoles are universal

The argument above has shown that almost all information concerning the detailed quantum state is unmeasurable by a semiclassical observer. This leaves open the possibility that the lowest $N^{1/4}$ multipoles can be measured. Here we will argue that even these measurements cannot be done with sufficient precision to distinguish between most microstates.

This is because almost all half BPS states of the kind that we are considering lie very close to a “typical state” [2]. Thus, as described in Sec. 2.1, almost all half BPS states with a bound on individual fermion excitation energies lie close to the curve

$$f(x) = \left(1 + \frac{N_c}{N}\right) x. \quad (49)$$

(Recall the continuum notation (7).) Equivalently, this is the typical spectrum of fermion energies in half-BPS states for which $0 \leq f_1 < f_2 < \dots < f_N \leq N_c + N$. We will show that the standard deviation to mean ratio of the moments M_k (2) is small in such an ensemble of states, implying that the differences between states cannot be observed by a semiclassical observer.

First define the ensemble of integers

$$\mathcal{E}_1 = \{f_i \mid 0 \leq f_1 < f_2 < f_3 \dots < f_N \leq N_c + N\} \quad (50)$$

The standard deviation to mean ratio of the moments (2)

$$\frac{\sigma(M_k)_{\mathcal{E}_1}}{\langle M_k \rangle_{\mathcal{E}_1}} \quad (51)$$

is computed by averaging M_k and M_k^2 over the ensemble (50), i.e. $\sigma(M_k)_{\mathcal{E}_1}^2 = \langle M_k^2 \rangle_{\mathcal{E}_1} - \langle M_k \rangle_{\mathcal{E}_1}^2$. The inequalities between the f_i make these averages cumbersome. Thus, we define a slightly different ensemble

$$\mathcal{E}_2 = \{f_i \mid 0 \leq f_1 \leq f_2 \leq f_3 \dots \leq f_N \leq N_c + N\} \quad (52)$$

in which the integers are allowed to be equal. Averages in this ensemble will turn out to be easier to compute. First we demonstrate the bound

$$\frac{\sigma(M_k)_{\mathcal{E}_1}}{\langle M_k \rangle_{\mathcal{E}_1}} < \frac{\sigma(M_k)_{\mathcal{E}_2}}{\langle M_k \rangle_{\mathcal{E}_2}}. \quad (53)$$

To show this bound consider the auxiliary integral

$$I = N(N + N_c)^k \int_0^1 x^k dx. \quad (54)$$

Sums of the form $\sum_i f_i^k$ with f_i drawn from either \mathcal{E}_1 or \mathcal{E}_2 can be considered as discrete approximations to this integral. In the $N \rightarrow \infty$ limit, an average over either \mathcal{E}_1 or \mathcal{E}_2 will converge to (54) via the results in the standard Monte Carlo theory of computing integrals (see Sec. 3 of [9]). Thus, as $N \rightarrow \infty$ the mean of M_k taken in either ensemble will be the same. Now observe that every set of f_i that appears in the ensemble (50) also appears in (52). However, (52) contains additional sets of integers in which some f_i coincide. But sets of coinciding f_i provide poorer approximations to the integral (54). Thus the variance of M_k computing in the second ensemble (52) must be larger. This shows the bound (53).

In the large N continuum limit, expectation values in the ensemble (52) can be computed by turning the sums over integers into integrals.¹ We should find that $\langle 1 \rangle_{\mathcal{E}_2} = 1$ and to this end we compute the normalization constant

$$C = \sum_{r_N=0}^{N_c+N} \sum_{r_{N-1}=0}^{r_N} \dots \sum_{r_1=0}^{r_2} 1 \approx \int_0^{N_c+N} dr_N \int_0^{r_N} dr_{N-1} \dots \int_0^{r_2} dr_1 = \frac{(N_c + N)^N}{N!}. \quad (55)$$

When $f(r_1, \dots, r_N)$ is symmetric in its arguments, we have the useful identity

$$\int_0^{N_c+N} dr_N \int_0^{r_N} dr_{N-1} \dots \int_0^{r_2} dr_1 f = \frac{1}{N!} \int_0^{N_c+N} dr_N \int_0^{N_c+N} dr_{N-1} \dots \int_0^{N_c+N} dr_1 f. \quad (56)$$

¹The reader may wonder why the same continuum approximation does not apply to the ensemble (50), obviating the need for the bound (53). When $N_c \gg N$, averages in (50) can indeed be approximated in this way. However, when N_c is $O(N)$ the continuum limit is more subtle and (53) is necessary.

Using this we can compute the mean

$$\langle M_k \rangle_{\mathcal{E}_2} = \frac{1}{C} \int_0^{N_c+N} dr_N \int_0^{r_N} dr_{N-1} \dots \int_0^{r_2} dr_1 M_k = \frac{N(N_c+N)^k}{k+1} \quad (57)$$

It is straightforward to compute $\langle M_k^2 \rangle_{\mathcal{E}_2}$ similarly, and this gives the standard deviation to mean ratio as

$$\frac{\sigma(M_k)_{\mathcal{E}_2}}{\langle M_k \rangle_{\mathcal{E}_2}} = \frac{k}{\sqrt{N(2k+1)}}. \quad (58)$$

This vanishes for small k , and is of order one when $k \sim N$. Using the bound (53) we can conclude that in the semiclassical $N \rightarrow \infty$ limit, almost all half BPS states have essentially identical low moments. Therefore the corresponding classical solutions have essentially identical low order multipoles, and the differences will not be observable by a semiclassical observer.

5 Discussion

We showed that the multipole expansion of half BPS asymptotically $\text{AdS}_5 \times S^5$ spacetimes encodes the tower of commuting, conserved charges that completely identifies the underlying quantum eigenstate. We then argued that a semiclassical observer, having access to coarse-grained observables, would only be able to measure the low multipoles, and that these were essentially universal for almost all half-BPS states. Thus the semiclassical observer necessarily loses information. The basis of the latter argument was that features of spacetime that occur at the Planck scale are only accessible in quantum gravity, and not to classical observers. This raises a question as to why the multipole expansion of the spacetime metric that we studied is itself reliable at high orders. In fact, strictly speaking, it is not classically reliable at very high orders – in quantum gravity there is a wavefunction over metrics that will lead to significant fluctuations in the precise form of the highly suppressed higher order multipoles. However, because we are constructing spacetimes dual to exact eigenstates of the moment operators, upon quantization the eigenvalues should still be extractable from the wavefunction of spacetime.

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A Asymptotic form of the 10D metric

Having computed the asymptotic expansion of the function u as in Sec. 3.1, one can find the multipole expansion of every component of the metric. The complete metric is given in (14,15,16,17). A multipole expansion for V_i (16) can be calculated as was done for u in section 3.1. The answer is

$$V_1(r, \varphi, \theta) = -2 \sin \varphi \sin \theta \left(\frac{\hbar N}{\rho^3} + \sum_{l=1}^{\infty} \frac{\hbar^{l+1} \sum_{f \in \mathcal{F}} A^l(f)}{\rho^{2l+3}} (-1)^l (l+1) \cdot \right. \\ \left. \cdot \left[{}_2F_1(-l, l+2, 1; \sin^2 \theta) + l {}_2F_1(1-l, l+2, 2; \sin^2 \theta) \right] \right). \quad (59)$$

The second component is given by $V_2 = -\cot \varphi V_1$. With these tools it is now possible to compute the components of the metric to any order. The first terms of V_i are

$$V_1 = -\frac{2 \sin \varphi \sin \theta}{\rho^3} \left(\hbar N - \frac{2\hbar^2(2-3\sin^2 \theta)(2E+N)}{\rho^2} \right. \\ \left. + \frac{6\hbar^3(3-12\sin^2 \theta+10\sin^4 \theta)(2M_2+2E+N)}{\rho^4} + \mathcal{O}\left(\frac{1}{\rho^6}\right) \right), \quad (60)$$

$$V_2 = -\cot \varphi V_1 \quad (61)$$

The first terms in the scalar functions in the metric are

$$R^2 = \frac{\rho^2}{\sqrt{2\hbar N}} \left(1 + \hbar \frac{(2\frac{E}{N}+1)(1-3\sin^2 \theta) - \cos^2 \theta N}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^4}\right) \right), \\ \tilde{R}^2 = \sqrt{2\hbar N} \cos^2 \theta \left(1 - \hbar \frac{(2\frac{E}{N}+1)(1-3\sin^2 \theta) - \cos^2 \theta N}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^4}\right) \right), \\ h^{-2} = R^2 + \tilde{R}^2 = \frac{\rho^2}{\sqrt{2\hbar N}} \left(1 + \hbar \frac{\cos^2 \theta + (2\frac{E}{N}+1)(1-3\sin^2 \theta)}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^4}\right) \right), \\ h^2 = \frac{\sqrt{2\hbar N}}{\rho^2} \left(1 - \hbar \frac{\cos^2 \theta + (2\frac{E}{N}+1)(1-3\sin^2 \theta)}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^4}\right) \right).$$

These quantities are more directly observable than the function u , but due to their cumbersome nature it is easier to work with u . Taking the semiclassical limit as in Sec. 3.2, we again see that at each order of the expansion a new moment M_i appears, as was expected from the behavior of u .

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